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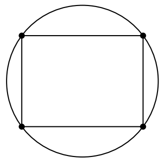
2601. Find the area enclosed by the parametric curve $x = 2 + t$, $y = 4 - t^2$, where $t \in \mathbb{R}$, and $y = 0$.

2602. A circle has a rectangle inscribed, whose sides have lengths in the ratio $1 : k$.



Show that the ratio of the areas of the rectangle and the circle is given by $4k : \pi\big(1 + k^2\big)$.

2603. Give the largest possible domain for the function

$$f(x) = \frac{1}{x^2 - |x|}.$$

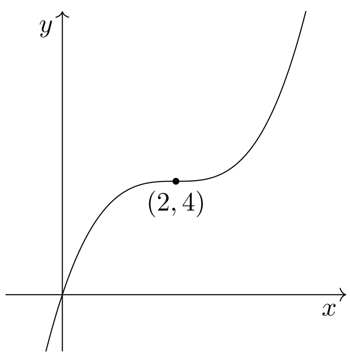
2604. A pseudo-random number generator is designed to produce numbers following the normal distribution $Z \sim N(0, 1)$. A programmer suspects that there is a bug causing the mean to be higher than this. A subsequent sample of 100 data yields $\bar{z} = 0.253$.

(a) Write down hypotheses for a suitable test.

(b) Carry out the test at the 5% significance level, stating your conclusions carefully.

2605. Show that $\displaystyle\int_{-1}^{1} \frac{2}{4 - x^2}\, dx = \ln 3$.

2606. A cubic graph $y = f(x)$ passes through O, and has the point $(2, 4)$ as a stationary point of inflection.



(a) Show that f has the form $f(x) = p(x - q)^3 + r$, where p, q, r are to be determined.

(b) Find f in expanded polynomial form.

2607. State, with a reason for each, the domain of

(a) $\arcsin(x)$,

(b) $\arccos(2x + 1)$.

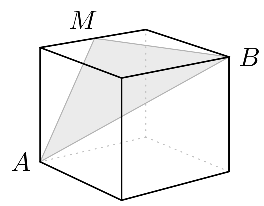
2608. A question asks for the solution of $\sin 2\theta = \frac{\sqrt{3}}{2}$, for $\theta \in [0, 180)$. A student writes the following:

$$\begin{aligned} &\sin 2\theta = \tfrac{\sqrt{3}}{2} \\ \Longrightarrow\ &2\theta = 60^\circ \\ \therefore\quad &\theta = 30^\circ, 150^\circ. \end{aligned}$$

Explain the error, and correct it.

2609. Two dice, one with twelve sides and the other with n, where $n < 12$, are rolled together. Determine the probability that the score on the n-sided die is larger than that on the twelve-sided die.

2610. The diagram shows a cube of side length l, with a triangle formed of two vertices and a midpoint.



Show that the area of triangle AMB is $\frac{\sqrt{6}}{4}l^2$.

2611. Solve $\sec^2 x + \sec x = 2$ for $x \in [0, 2\pi)$.

2612. A jeep of mass 500 kg is towing a broken-down car of the same mass with a short, light rope. The car has no brakes. Resistances of 600 N and 1450 N act on jeep and car respectively. They meet a steep downward slope of inclination 30° and length 10 m.

(a) Show that the two vehicles cannot descend at constant speed without the rope going slack.

(b) To keep the rope taut, the jeep must accelerate down the slope. Determine the car's minimum possible speed at the bottom, if the rope is to remain taut until this point.

2613. Sketch $y = x^5 - 8x^3 + 16x$. You don't need to find the coordinates of stationary points.

2614. Invertible functions f, g, h are such that

$$\begin{aligned} &gf^{-1}(a) = 0, \\ &gh(0) = a, \\ &h^{-1}f^{-1}(a) = 0. \end{aligned}$$

Show that $a = 0$.

2615. Write the following in terms of $\log_x y$:

(a) $\log_x \big(y^2\big)$,

(b) $\log_{x^2} \big(y^2\big)$,

(c) $\log_{x^2} y$.

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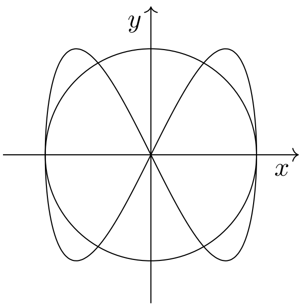
2616. Use the substitution $u = 2 + \sqrt{x}$ to show that

$$\int_0^1 \frac{2}{2+\sqrt{x}}\,dx = 4 + 8\ln\tfrac{2}{3}.$$

2617. A parametric curve is defined by the equations $x = \sin t$, $y = \sin 2t$, for $t \in (-\pi, \pi]$. This curve is shown below, with a unit circle centred at O.



Find the coordinates of the points of intersection of the curve and the circle.

2618. For events A and B with non-zero probabilities, and which are not mutually exclusive, prove the following statement:

$$\mathbb{P}(A \mid B) = \mathbb{P}(B \mid A) \iff \mathbb{P}(A) = \mathbb{P}(B).$$

2619. Show that the straight line through the points $(1, \cos^2 a)$ and $(\sin^2 a, 1)$ has gradient $-\tan^2 a$.

2620. Six dice are rolled. State, with a reason, which, if either, of the following events has the greater probability:

- there is exactly one five,
- there are no fives.

2621. By factorising, sketch the region or regions of the (x, y) plane which satisfy $xy - x + y - 1 < 0$.

2622. (a) Express $12 \div 5$ as the sum of an integer and a proper fraction.

(b) Express $(4x^2+x) \div (x+1)$ as the sum of a linear function and a proper algebraic fraction.

2623. A linear function g is such that, for some non-zero constant a,

$$\int_{-a}^{a} \mathrm{g}(x)\,dx = 0 \text{ and } \int_{a}^{3a} \mathrm{g}(x)\,dx = 4a.$$

Find $\mathrm{g}(a)$.

2624. A six-sided and a four-sided die are rolled, and the magnitude of the difference between the scores is designated X. Determine the probability that two consecutive simultaneous rolls yield the same value of X.

2625. You are given that

$$\frac{d}{dx}\left(x + \frac{d}{dx}(x+y)\right) = x.$$

Find $\dfrac{d^2y}{dx^2}$ in terms of x.

2626. The vertical speed v ms^{-1} of a skydiver, t seconds after a jump, is modelled by

$$\frac{dv}{dt} = g - 0.1v.$$

This is a separable differential equation.

(a) Interpret the terms g and $-0.1v$.

(b) Show that $v = 98 - 98e^{-0.1t}$.

(c) Find the exact time taken by the skydiver to reach half of terminal velocity.

2627. Explain whether each of the following statements, concerning a substitution $u = x+2$ and a function f, is true or false:

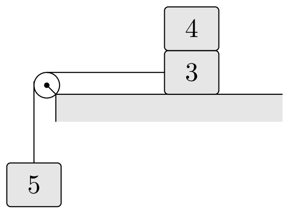
(a) $\mathrm{f}'(x) \equiv \mathrm{f}'(u)$,

(b) $\dfrac{d}{dx}\,\mathrm{f}(x) \equiv \dfrac{d}{du}\,\mathrm{f}(u)$,

(c) $\dfrac{d}{dx}\,\mathrm{f}(u) \equiv \mathrm{f}'(u)$.

2628. Solve $32\log_x 2 + \log_2 x = 12$.

2629. Two blocks of mass 3 and 4 kg are stacked on a table. The lower 3 kg block is connected by an inextensible string, which is passed over a smooth, light, fixed pulley, to a block of mass 5 kg which hangs freely. Both contacts with the lower block are rough, coefficient of friction $\frac{1}{2}$.



(a) State a further assumption needed, regarding the string, to allow solving of this system.

(b) Making this assumption, find

i. the acceleration of the system,

ii. the friction between the stacked blocks.

2630. You are given that, for constants a, b, the solution of the inequality $ax^2 + bx - a^3 \leq 0$ is

$$x \in (\infty, -1] \cup [1, \infty).$$

(a) Write down the value of b.

(b) Find a.

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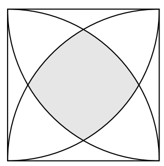
2631. Show that the equation $z = \sqrt{x^2+y^2}$ describes the curved surface of a cone.

2632. Express the following set as a list of elements, in the form $\{a_1, a_2, ..., a_n\}$.

$$\{x \in \mathbb{Z} : x^2 < 4\} \cap \{y \in \mathbb{Z} : y^2 + y > 4\}.$$

2633. Let $X_1 \sim B(n, p_1)$ and $X_2 \sim B(n, p_2)$ be a pair of independent binomial variables, with $p_1 \neq p_2$. State, with a reason, whether the sum $X_1 + X_2$ is binomially distributed.

2634. A pattern is constructed from a unit square and four quarter circles, as depicted below:



Show that the area shaded is $\frac{\pi}{3} + 1 - \sqrt{3}$.

2635. A *minimum bounding rectangle* is defined to be the smallest rectangle, with sides parallel to the x and y axes, which contains a set of objects.

Show that the minimum bounding rectangle for the loci $x^2 + 4x + y^2 + 6y = 3$ and $|x| + |y + 1| = 4$ is a square of side length 10.

2636. Show that $y = x$ is a tangent to the curve

$$y = \frac{1}{x-1} + \frac{1}{(x-1)^2}.$$

2637. Three cards are dealt from a standard deck.

(a) Show that the number of hands in which all the cards are of different suits is 8788.

(b) Find the number of hands in which there is one club, one diamond and one heart.

(c) Hence, given that all three are of different suits, find the probability that there are no spades.

(d) Comment on your result.

2638. A function f is defined as

$$x \longmapsto \frac{6x^4 + x^3 + 4x^2 + x - 4}{x^2 + 1}.$$

Show that f is not a polynomial function.

2639. (a) Show that the normal to $y = x^2$ at $x = p$ has equation $2py = 2p^3 + p - x$.

(b) Hence, determine which of the points $(3, 0)$ and $(0, 4)$ is closer to the parabola $y = x^2$.

2640. An equilateral triangle is placed with two vertices at $(0, 0)$ and $(2p, 0)$, where $p \in \mathbb{Z}$. Prove that the y coordinate of the third vertex cannot be an integer. You may assume that surds are irrational.

2641. Find $\int \sec^2 \frac{1}{3}x \, dx$.

2642. In the equation $\varepsilon x^2 - 2x + 10 = 0$, the coefficient ε is very close to zero.

(a) Explain why the equation must have a root close to $x = 5$.

(b) Show that $x = \dfrac{1 \pm \sqrt{1 - \frac{5}{2}\varepsilon}}{\varepsilon}$.

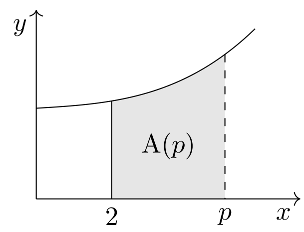
(c) Show that the positive square root gives a large root close to $x = \frac{2}{\varepsilon}$.

(d) Explain the error in the following argument: "With the negative square root, for small ε the numerator tends to zero, so there should be a root at around $x = 0$."

2643. State, with a reason, whether the following is true:

$$\frac{d(\cos 2\theta)}{dt} \times \frac{dt}{d\theta} \equiv -2\sin 2\theta.$$

2644. An area function, for the curve $y = 12x^3 + 10x + 2$, is defined as below for $p > 2$:



(a) Find $\mathrm{A}(p)$ in terms of p.

(b) Solve $\mathrm{A}(p) = 7390$.

2645. The two *conic sections* $(x-a)^2 + (y-b)^2 = r^2$ and $(x-a)^2 - (y-b)^2 = r^2$ are circular and hyperbolic respectively.

(a) Find the coordinates of any intersections.

(b) At these intersections, find the equations of the tangents to the hyperbola.

(c) Hence, show that the curves are tangent.

(d) The hyperbola has oblique asymptotes whose gradients are ± 1. Sketch both curves on one set of axes.

2646. One of the following statements is true; the other is not. Prove the one and disprove the other.

(a) $x^2(e^x - 2) = 0 \implies x = 0$,

(b) $x^2 e^{x-2} = 0 \implies x = 0$.

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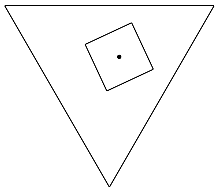
2647. 30% of motor insurance claims made to a company are bogus. Of bogus claims, only 40% are rejected as bogus, while 25% of genuine claims are rejected as bogus.

(a) Represent this information on a tree diagram.

(b) Find the probability that a claim is rejected.

(c) Find the probability that a rejected claim is, in fact, genuine.

2648. An equilateral triangle of side length 1 has a square of area A drawn inside it. This square can rotate freely around its centre without intersecting the triangle.



Show that $A \in (0, 1/6)$.

2649. Show that the points $(2,0)$, $(1,1)$, $(4,4)$ and $(5,1)$ all lie on the same circle.

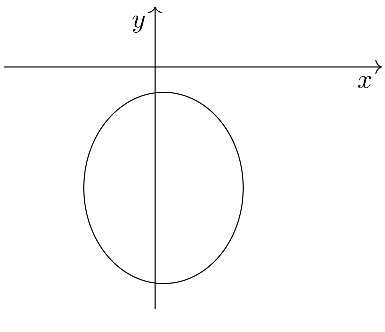
2650. Show that, if x and y are related by $y = a \cdot b^x$, then $\ln y$ and x are related linearly.

2651. If $\int_0^a f(x)\,dx = b$, find, in terms of a, b:

(a) $\int_0^{2a} f\left(\frac{1}{2}x\right) dx$,

(b) $\int_{-a}^{a} f\left(\frac{1}{2}(x+a)\right) dx$.

2652. The diagram below shows an ellipse:



The equation of the ellipse is

$$9x^2 - 6x + y^2 + 8y + 7 = 0.$$

Find the coordinates of the centre.

2653. Explain, with reference to a sketched velocity-time graph, why, during any period of time, there must be at least one point at which the instantaneous acceleration of a particle is equal to the average acceleration over the entire period.

2654. Find constants P, Q to make this an identity:

$$\frac{2}{x^6 + x^3} \equiv \frac{P}{x^3} + \frac{Q}{x^3 + 1}.$$

2655. Either prove or disprove the following statement: "If $f''(x) = f'(x)$, then $f'(x) = f(x)$".

2656. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning real numbers x and y:

① $y = \sin x$,

② $x = \arcsin y$.

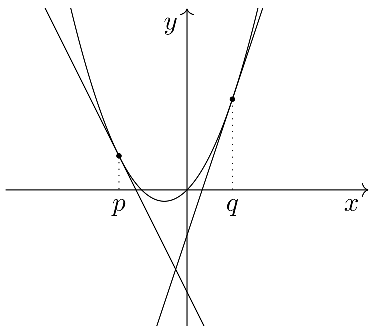
2657. Solve $\sqrt{3}\tan^2 x = 4\tan x - \sqrt{3}$, for $x \in [0, 2\pi)$.

2658. A projectile is fired from a height of $\sqrt{3}$ metres above horizontal ground, to hit a target at ground level, 5 metres away horizontally. It is fired at 30° above the horizontal, at speed u. Show that

$$u^2 = \frac{25g}{4\sqrt{3}}.$$

2659. Describe the single transformation which takes the graph $x = f(y)$ onto the graph $x = k - f(y)$.

2660. A parabola has equation $y = x^2 + kx$, for some constant k. Two tangent lines are drawn to the parabola at distinct points $x = p$ and $x = q$.



Prove that the tangents meet at $x = \frac{1}{2}(p + q)$.

2661. This question concerns $I = \int_1^2 e^x \sqrt{x}\,dx$.

(a) Estimate I with a four-strip trapezium rule.

(b) You are given that $y = e^x\sqrt{x}$ has no points of inflection for $x \in [1, 2]$. By evaluating $f''(1)$, determine whether your answer in (a) is an over or underestimate.

2662. Prove that, if two parabolae of the form $y = f(x)$ have three distinct points in common, then they must be the same parabola.

2663. Determine the mode of $B(6, 1/4)$.

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2664. Show that the curve $y = (2x+1)(2x+3)(2x+5)$, with the x axis, encloses two regions of equal area.

2665. In this question, ignore leap years.

Determine the smallest number of people such that, in any given week, the probability of at least one person having a birthday is over 50%.

2666. Solve $2^{3+2x} + 2^{1+x} = 1$.

2667. For a piece of equipment, efficiencies E at various operating temperatures $T°$ are given below.

$T°$	60	65	70	75	80
E	20	39	71	76	21

It is proposed that the relationship be modelled by

$$E = 20 + (T-60)^2\left(1 - 0.05\,(T-60)\right).$$

(a) Show that this is consistent with the data.

(b) Determine the optimal operating temperature in $[60°, 80°]$, as predicted by this model.

2668. In this question, do not use a calculator.

The equation $\mathrm{f}(x) = 75x^3 - 35x^2 - 8x + 4 = 0$ has precisely two roots, both of which are rational.

(a) Explain how you know that the graph $y = \mathrm{f}(x)$ has a stationary point on the x axis.

(b) Using calculus, determine the solution of the equation.

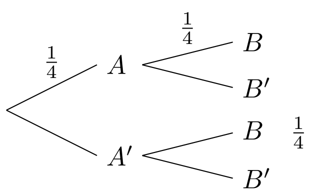
2669. Write down the ranges of the following functions, when they are defined over the largest possible real domains:

(a) $x \mapsto \dfrac{1}{x^2+1}$,

(b) $x \mapsto \dfrac{1}{x^3+1}$,

(c) $x \mapsto \dfrac{1}{x^4+1}$.

2670. A tree diagram is drawn to represent probabilities of events A and B as follows:



Determine the following probabilities:

(a) $\mathbb{P}(B)$,

(b) $\mathbb{P}(A \mid B)$.

2671. This question concerns a limit L, which is defined in terms of $x > 0$ as follows:

$$L = \lim_{p \to 0} \frac{\sqrt{x^2+p} - x}{p}.$$

(a) Show that $\sqrt{x^2+p} \equiv x\left(1 + \dfrac{p}{x^2}\right)^{\frac{1}{2}}$.

(b) Using the binomial expansion, show that

$$\left(1 + \frac{p}{x^2}\right)^{\frac{1}{2}} \equiv 1 + \frac{p}{2x^2} - \frac{p^2}{8x^4} + \ldots$$

You may assume that the sum converges.

(c) Hence, find L in terms of x.

2672. Using $\cos 2\theta \equiv 2\cos^2\theta - 1$, find

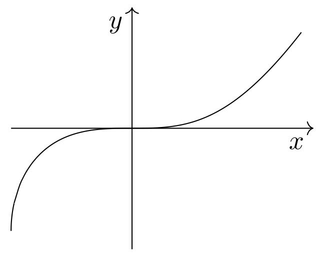
$$\int \cos^2\left(4x + \tfrac{\pi}{12}\right)\,dx.$$

2673. Functions f and g have instructions

$$\mathrm{f}(x) = \log_{10}\left(x^2\right),$$
$$\mathrm{g}(x) = 2\log_{10} x.$$

Explain carefully, with reference to domains, why f and g do not necessarily produce the same output.

2674. Determine whether the equation $y = \sqrt{x^3+2} - k$, for some constant k, could generate the following graph:



2675. Solve $\left(e^{2x}-1\right)^4 + \left(e^{2x}-1\right)^2 = 0$.

2676. Either prove or disprove the following: "If three forces of magnitude P, $2P$, $3P$ act on an object, then that object remains in equilibrium if and only if the three forces have the same line of action."

2677. A *Kepler triangle* is a right-angled triangle, whose edge lengths are in geometric progression. Show that the ratio of the progression is $\sqrt{\phi}$, where ϕ is the golden ratio $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$.

2678. Parametric equations are given as

$$x = \sec t,$$
$$y = \sin t + \cos t.$$

Verify that these satisfy the Cartesian equation

$$x^2(y^2-1) = 2(xy-1).$$

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2679. The variables $X_1, ..., X_n$, which model a sample of size n, are independently distributed according to $X_i \sim N(\mu, \sigma^2)$. Write down the distributions of

(a) $\bar{X}$

(b) $X_i + 2$

(c) $\displaystyle\sum_{i=1}^{n} X_i$.

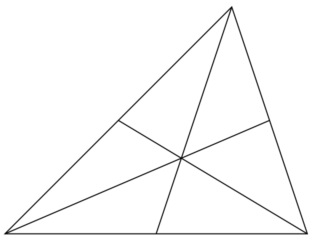
2680. Two parabolae are given as

$$y - x^2 + x = 0,$$
$$x - y^2 + y = 0.$$

(a) Show that the parabolae are tangent.

(b) Sketch them on a single set of axes.

2681. The *medians* of a triangle are the lines joining the vertices to the midpoints of their opposite sides. All three medians meet at one point, known as the *centroid*. The centroid divides each median in the ratio 1 : 2.



Prove that the medians divide a triangle up into six smaller triangles of equal area.

2682. Two students are working on the integral

$$I = \int \frac{1}{2x}\, dx.$$

① The first student took out a factor of $\frac{1}{2}$, and got $I = \frac{1}{2} \ln |x|$.

② The second student used the reverse chain rule, and ended up with $I = \frac{1}{2} \ln |2x|$.

They cannot work out who is right and who is wrong. Resolve the issue.

2683. Using calculus, or otherwise, find the range of

$$x \longmapsto \frac{1}{\sin^2 x + \sin x + 1}.$$

2684. Solve $(x^2 - 1)^5 + (x^2 - 1)^4(2x - 3) = 0$.

2685. A polynomial function f is invertible over each of the domains $(-\infty, k]$ and $[k, \infty)$, but not over $\mathbb{R}$. Prove that $y = \mathrm{f}(x)$ is stationary at $x = k$.

2686. Three dice have been rolled, giving scores X, Y, Z. Determine whether the fact "$X = Y$" increases, decreases or doesn't change $\mathbb{P}(X + Z) = 7$.

2687. A hyperbola and a circle have equations

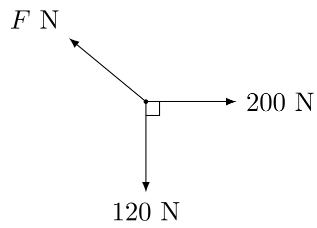
$$xy = 2,$$
$$x^2 + y^2 = 5.$$

(a) Find the coordinates of any intersections.

(b) On a sketch, shade the regions which satisfy

$$(xy - 2)(x^2 + y^2 - 5) \leq 0.$$

2688. An object of mass 50 kg has forces acting on it as shown in the diagram. The object accelerates at 2.6 ms^{-2}, parallel to the force of magnitude F N.



(a) Show that the force of magnitude F acts at an angle $\pi - \arctan \frac{3}{5}$ to the 200 N force.

(b) Find F to 3sf.

2689. Use integration by parts to show that

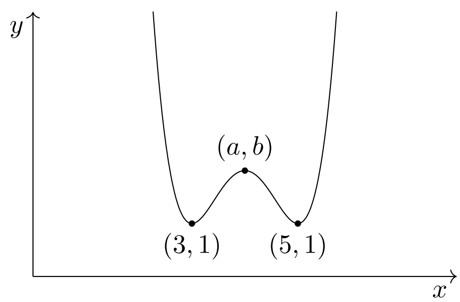
$$\int_0^{\frac{\pi}{2}} x \sin x \, dx = 1.$$

2690. Either prove or disprove the following statement: "A set of three linear equations in two unknowns cannot have a unique solution."

2691. Explain and correct the error in the following:

$$\ln(x^2 + y - 1) = \ln x + \ln y$$
$$\Longrightarrow x^2 + y - 1 = x + y.$$

2692. A monic quartic graph has local minima at $(3, 1)$ and $(5, 1)$, and a local maximum at (a, b).



(a) Show that the graph must have the form

$$y = (x - p)^2(x - q)^2 + r,$$

where p, q, r are constants to be determined.

(b) Find the local maximum (a, b).

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FEEDBACK: GILES.HAYTER@WESTMINSTER.ORG.UK

2693. Prove that, for every quadratic function f, there are two domains D_1 and D_2, with $D_1 \cup D_2 = \mathbb{R}$, over which f is invertible.

2694. A particular rare disease affects 1 in 10000 of the population. There is an accurate test for it. The test registers positive in 99% of cases in which a person has the disease, and registers negative in 98% of cases in which they don't.

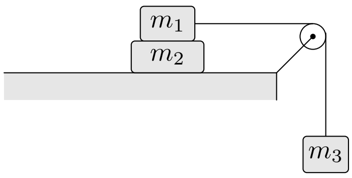
(a) Construct a tree diagram to represent this.

(b) A person is tested at random, and tests +ve. Find the probability they have the disease.

(c) Comment on your result.

2695. Sketch $y = \operatorname{arcsec} x$, marking the coordinates of any major features.

2696. A pulley system is set up on a table as depicted. The pulley is light and smooth, and the string is light and inextensible. The coefficient of friction at both surfaces of the lower stacked block is μ.



Show that, whatever the values of m_1, m_2, m_3 and μ, the lower stacked block will not accelerate.

2697. Equations $\mathrm{f}(x) = 0$ and $\mathrm{g}(x) = 0$, where f and g are polynomial functions, have the same solution set S. The equation $\mathrm{f}(x) = \mathrm{g}(x)$ is denoted E. State, with a reason, whether the following claims hold:

(a) "E has solution set S",

(b) "the solution set of E contains S",

(c) "the solution set of E is a subset of S".

2698. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$|x + y| < 1, \qquad |x - y| < 1.$$

2699. Prove the following trigonometric identity:

$$\cos x \cos 2x + \sin x \sin 2x \equiv \cos x.$$

2700. By substitution, find and simplify $\displaystyle\int \frac{2x^2}{x+1}\,dx$.

——— End of 27th Hundred ———

WWW.GILESHAYTER.COM/FIVETHOUSANDQUESTIONS.ASP
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